

# INTERESTING EXAMPLES IN $\mathbb{C}^2$ OF MAPS TANGENT TO THE IDENTITY WITHOUT DOMAINS OF ATTRACTION

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**ABSTRACT.** We give an interesting example of a map in  $\mathbb{C}^2$  that is tangent to the identity, but that does not have a domain of attraction along any of its characteristic direction. This map has three characteristic directions, two of which are not attracting while the third attracts points to that direction, but not to the origin. In addition, we show that if we add higher degree terms to this map, sometimes a domain of attraction along one of its characteristic directions will exist and sometimes one will not.

## INTRODUCTION

In this article, we give an example of a map tangent to the identity in  $\mathbb{C}^2$  that exhibits new and interesting dynamical behavior. We demonstrate that there are no domains of attraction to the origin along any of its characteristic directions. In addition, we show that one of the directions attracts points to itself, but not to the origin. Consequently, we will see that within any arbitrarily small open neighborhood of the origin, there are points (an open subset of  $\mathbb{R}^2$ ) that are not fixed and that, under iteration, remain close to the origin but do not converge to the origin. The map  $f$  we discuss here, as far as the author knows, is the first example of a map that is tangent to the identity that is shown to exhibit this interesting behavior.

The additional examples we discuss are modifications of this main example and are created by adding higher degree terms to the main example. We show that this addition of higher degree terms can lead to interesting changes in the dynamics near the origin. The following theorem is about our main example.

**Theorem 1.** *Let  $f(z, w) = (z[1 - (z - w)], w[1 + (z - w)])$ . Then  $f$  has the following properties:*

- (1)  *$f$  has characteristic directions  $[1 : 0]$ ,  $[0 : 1]$ ,  $[1 : 1]$ ;*
- (2) *the complex line  $\{z = w\}$ , which corresponds to  $[1 : 1]$ , is fixed pointwise;*
- (3) *there is a domain  $A \subset \mathbb{R}^2 \subset \mathbb{C}^2$  with the origin in its boundary whose points converge to the real line  $\{z = w\}$ , but do not converge to the origin;*
- (4) *for  $(z, w) \in A$ ,  $\lim_{n \rightarrow \infty} n|z_n - w_n| < 1$ ; and*
- (5) *there is no domain of attraction to the origin.*

Note that the convergence of  $z_n - w_n$  to 0 might be significantly faster than the bound  $n^{-1}$  that is given in part (4).

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*Remark.* In this article, it is sometimes more convenient to perform a linear conjugation of  $f$  that sends  $[1 : 1]$  to  $[1 : 0]$ . This linear conjugation is discussed in (3.2). The new expression for  $f$  is:

$$(0.1) \quad \tilde{f}(x, y) = (x - y^2, y - xy) \quad \text{with } [1 : 0] \text{ degenerate characteristic direction.}$$

We find this way of expressing  $f$  particularly useful in §5, where we add higher degree terms and study when a domain of attraction exists. In §5, we show that some choices of higher degree terms can cause a domain of attraction to the origin to exist while other choices cause one not to exist. We use [L3, Theorem A], which can be found in §5, to see that there are many choices of higher degree terms that can lead to a domain of attraction along  $[1 : 0]$ , but not along its other characteristic directions. The following theorem is a (significant) simplification of [L3, Theorem A] to maps whose lower degree terms are of the form (0.1).

**Theorem 2.** *Let  $g(x, y) = (x - y^2 + ax^{r+1}, y - xy)$ , where: (1)  $a \in \mathbb{C}^\times$  and  $r \in \mathbb{N}_{\geq 3}$ ; or (2)  $a \notin \mathbb{R}_{\geq 0}$  and  $r = 2$ . Then  $g$  has a domain of attraction to the origin along  $[1 : 0]$ .*

More generally, again by simplifying [L3, Theorem A], the previous theorem holds for  $g + \hat{g}$ , where  $\hat{g}(x, y) = (yO((x, y)^2) + O(x^{r+2}), yO((x, y)^2) + O(x^{r+2}))$ .

We also show that some choices of higher degree terms cause there to be no domain of attraction to the origin along any characteristic direction.

**Theorem 3.** *Let  $h(x, y) = (x - y^2 + ax^3, y - xy)$ . If  $a \in \mathbb{R}_{>0}$ , then  $h$  has no domain of attraction to the origin along any direction.*

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## 1. THE MAIN EXAMPLE

Let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be the map:

$$f(z, w) = (z + p(z, w), w + q(z, w)) = (z(1 - (z - w)), w(1 + (z - w))).$$

Then  $f$  is tangent to the identity (i.e.,  $df|_{(0,0)} = \text{Id}$ ). Characteristic directions of  $f$  are the directions fixed by  $(p, q)$ . Equivalently, the characteristic directions of  $f$  are given by the zeros of:

$$(1.1) \quad r(z, w) = zq(z, w) - wp(z, w) = 2zw(z - w).$$

In particular,  $f$  has exactly three characteristic directions:  $[1 : 0]$ ,  $[0 : 1]$ , and  $[1 : 1]$ . The directions  $[1 : 0]$  and  $[0 : 1]$  are both non-degenerate because  $(p, q)(1, 0) \neq (0, 0)$  and  $(p, q)(0, 1) \neq (0, 0)$ . Since both of these directions are non-degenerate, they have corresponding directors.

The director of  $[1 : 0]$  is defined to be:

$$\left[ \frac{1}{p(1, w)} \frac{d}{dw} r(1, w) \right] \bigg|_{w=0} = -2 < 0.$$

Similarly, the director of  $[0 : 1]$  is:

$$\left[ \frac{1}{q(z, 1)} \frac{d}{dz} (-r(z, 1)) \right] \bigg|_{z=0} = -2 < 0.$$

Since the directors of  $[1 : 0]$  and  $[0 : 1]$  are both negative, by [AR, Corollary 8.11] there is no domain of attraction whose points converge to the origin along  $[1 : 0]$  or  $[0 : 1]$ .

If the orbit of a point converges to the origin along a direction, that direction must be a characteristic direction by [H1, Proposition 1.3]. Hence, the only directions along which points could converge to the origin are  $[1 : 0]$ ,  $[0 : 1]$ , and  $[1 : 1]$ . We just showed that there is no domain of attraction along  $[1 : 0]$  or  $[0 : 1]$ , however some points are attracted to the origin along those directions. In particular, on  $\{w = 0\}$ :

$$f(z, 0) = (z(1 - z), 0)$$

attracts the cauliflower set  $\mathcal{C} \times \{0\} \subset \mathbb{C} \times \{0\}$  to the origin along  $[1 : 0]$ , where:

$$\mathcal{C} := \text{interior of } \{z \in \mathbb{C} \mid f(z) := z(1 - z), \lim_{n \rightarrow \infty} |f^n(z)| \not\rightarrow \infty\}.$$

Similarly, on  $\{z = 0\}$ ,

$$f(0, w) = (0, w(1 - w)),$$

attracts  $\{0\} \times \mathcal{C}$  to the origin along  $[0 : 1]$ . Hence, under iteration by  $f$ , the sets  $f^{-k}(\mathcal{C} \times \{0\})$  and  $f^{-k}(\{0\} \times \mathcal{C})$  converge to the origin along  $[1 : 0]$  and  $[0 : 1]$ , respectively, for all  $k \in \mathbb{N}$ . So there are complex curves, but not domains, that are attracted to the origin along those two characteristic directions. In §3, we analyze the dynamics of  $f$  along the remaining characteristic direction  $[1 : 1]$ , but first, in §2, we visualize the dynamics of  $f$  in  $\mathbb{R}^2$ .

## 2. ILLUSTRATING THE DYNAMICS OF $f$ IN $\mathbb{R}^2$

To more easily visualize  $f$ , we temporarily restrict the domain of  $f$  to  $\mathbb{R}^2$ . From the previous section, we know the dynamics along the axes and their preimages:

$$\begin{aligned} \bigcup_{n \geq 0} f^{-n}(\{z = 0\}) &= \{z = 0\} \cup \{z - w = 1\} \cup \{(z - w)(1 - z - w) = 1\} \cup \bigcup_{k \geq 3} f^{-k}(\{z = 0\}) \\ \bigcup_{n \geq 0} f^{-n}(\{w = 0\}) &= \{w = 0\} \cup \{z - w = -1\} \cup \{(z - w)(1 - z - w) = -1\} \cup \bigcup_{k \geq 3} f^{-k}(\{w = 0\}). \end{aligned}$$

Since  $\mathcal{C} \cap \mathbb{R} = (0, 1)$ , points in  $I_1 = \{(0, 1) \times \{0\}\}$  and  $I_2 = \{\{0\} \times (0, 1)\}$  (and their preimages) are attracted to the origin along  $[1 : 0]$  and  $[0 : 1]$ , respectively. The endpoints of  $I_1$  and  $I_2$  (and their preimages) are preimages of the origin. The curve segments given by  $I_1, I_2$  and their preimages are depicted in Figure 1.

Figure 2 was drawn using Dynamics Explorer<sup>a</sup>. The characteristic direction  $[1 : 1]$  corresponds to the line  $\{z = w\}$ . In Figure 2, points are colored blue if they converge to the line  $\{z = w\}$ . The boundary of the blue region is the collection of curves in Figure 1 given by the preimages of  $I_1$  and  $I_2$ . In Figure 2, points are colored pale red if they do not converge to  $\{z = w\}$  or to the origin. The dark red indicates a change in behavior that should appear as a line of blue (converge to  $\{z = w\}$ ) or yellow (converge to  $(0, 0)$ ), however the solitary curve was too thin to show up in a different color than the surrounding curves. In particular, preimages of  $\{z = w\}$  converge to  $\{z = w\}$  and show up as the dark red lines that enter the blue region (they correspond to  $\{z = w\}$ ,  $f^{-1}(\{z = w\}) = \{z + w = 1\} \cup \{z = w\}$ , etc.). Two of these lines are labeled in blue with boxes around their equations. Similarly, the boundary of the blue region, comprised of preimages of  $I_1$  and  $I_2$ , should be yellow as these points converge to  $(0, 0)$ , but the curve is too thin to appear.

<sup>a</sup>Dynamics Explorer is a tool for exploring dynamical systems that was written by Suzanne and Brian Boyd. It is available for download here: <http://sourceforge.net/projects/detool/>.

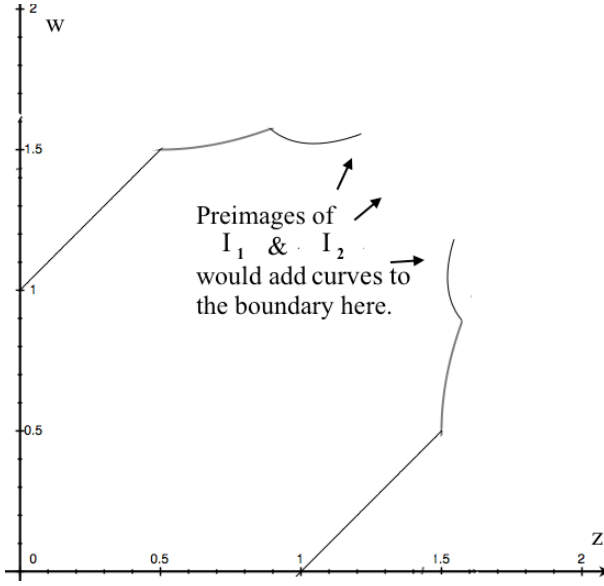


FIGURE 1.  
Boundary: Preimages in  $\mathbb{R}^2$  of  
 $I_1 \subset \{w = 0\}$  &  $I_2 \subset \{z = 0\}$ .

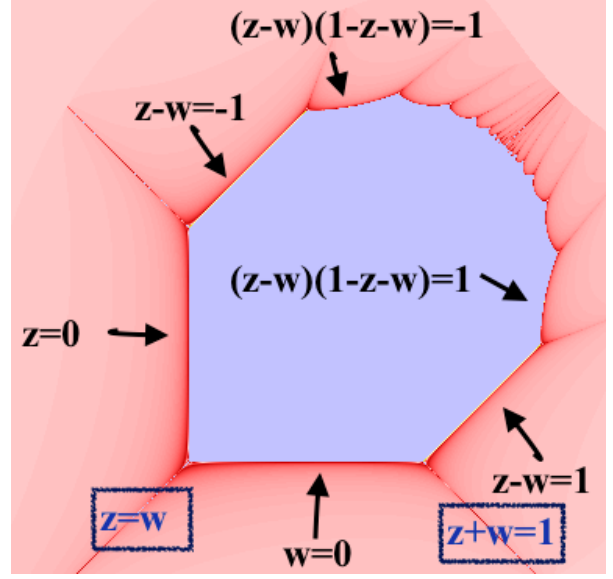


FIGURE 2.  
Dynamical Picture in  $\mathbb{R}^2$ : Blue  
points converge to  $\{z = w\}$ .

### 3. DYNAMICS OF $f$ NEAR $[1 : 1]$

The line  $\{z = w\}$  is point-wise fixed (since  $f(z, z) = (z, z)$ ) and corresponds to the direction  $[1 : 1]$ . This direction is a degenerate characteristic direction of  $f$  since  $(p, q)(1, 1) = 0(1, 1) = (0, 0)$ . It is easy to check that  $[1 : 1]$  is an apparent characteristic direction, whereas  $[1 : 0]$  and  $[0 : 1]$  are both fuchsian characteristic directions (definitions in [V]).

In  $\mathbb{C}^2$ , for a map tangent to the identity, there are many results on whether a domain of attraction exists along a given characteristic direction that depend, in part, on if the direction is apparent, fuchsian, irregular, or dicritical (see [AR, H1, H2, L1, L2, L3, V]). In [V], Vivas showed that  $f$  has no domain of attraction to the origin along  $[1 : 1]$ . We expand on this to show that: (1) the dynamics near  $[1 : 1]$  is very interesting and (2)  $f$  actually has no domain of attraction to the origin. Now we pay more rigorous attention to the interesting dynamical behavior depicted in Figure 2. In particular, we prove there is an open subset of  $\mathbb{R}^2$  whose points converge to  $\{z = w\}$ .

For the remainder of this section, we restrict the domain of  $f$  to  $\mathbb{R}^2$ . We first show that

$$(3.1) \quad A := \{(z, w) \in \mathbb{R}^2 \mid z > 0, w > 0, z + w < 1\}$$

is  $f$ -invariant. By looking at where the lines  $z = 0$ ,  $w = 0$  and  $z + w = 1$  intersect, we see that for  $(z, w) \in A$ ,  $-1 < z - w < 1$ . For any  $(z, w) \in A$ , with  $(z_1, w_1) := f(z, w)$ :

$$\begin{aligned} z_1 &= z(1 - (z - w)) > z \cdot 0 = 0, \\ w_1 &= w(1 + (z - w)) > w \cdot 0 = 0, \text{ and} \\ z_1 + w_1 &= z + w - (z - w)^2 \leq z + w < 1. \end{aligned}$$

Hence,  $A$  is  $f$ -invariant.

Next we show that points inside  $A$  converge to  $\{z = w\}$ . Let  $l(z, w) = (z + w, z - w) := (x, y)$  and conjugate  $f$  by  $l$ . Then  $f$  acts on the new coordinates by:

$$(3.2) \quad (x, y) \mapsto (x_1, y_1) := l \circ f \circ l^{-1}(x, y) = (x - y^2, y(1 - x)) \quad \text{and} \quad [1 : 1] \text{ moves to } [1 : 0].$$

For  $(z, w) \in A$ , showing that  $(z_n, w_n)$  converges to  $\{z = w\}$  is equivalent to showing that  $y_n \rightarrow 0$  or  $|y_n|^{-1} \rightarrow \infty$ . We change coordinates again by defining  $v := y^{-1}$ . Then  $f$  sends  $v$  to:

$$v_1 = \frac{v}{1 - x} = v \left( 1 + x + \sum x^j \right).$$

In  $A$  with  $z \neq w$ , we have  $1 > x = z + w \geq |z - w| = |v|^{-1} > 0$ , so:

$$|v_1| > |v|(1 + x) > |v|(1 + |v|^{-1}) = |v| + 1 \quad \text{and} \quad |v_n| > |v| + n.$$

Hence,  $\lim_{n \rightarrow \infty} |z_n - w_n| = \lim_{n \rightarrow \infty} |v_n|^{-1} = 0$  and all points inside  $A$  converge to  $\{z = w\}$ . In addition, the rate of convergence of  $z_n - w_n$  to 0 is at least  $n^{-1}$  since  $|z_n - w_n| < n^{-1}$  for all  $n > 0$ .

Lastly, we show that points in  $A$  do not converge to the origin. Observe that for any  $(z, w) \in A$ ,

$$\begin{aligned} \text{if } z - w \leq 0, \quad \text{then } z_1 - w_1 &= (z - w)(1 - z - w) \leq 0; \text{ and} \\ \text{if } w - z \leq 0, \quad \text{then } w_1 - z_1 &= (w - z)(1 - z - w) \leq 0. \end{aligned}$$

Take any  $(z, w) \in A$  with  $z - w \leq 0$ . Then

$$\frac{z_n}{z} = \prod \frac{z_{j+1}}{z_j} = \prod (1 - (z_j - w_j)) \geq 1$$

since  $z - w \leq 0$  implies that  $z_j - w_j \leq 0$  for all  $j$ . Similarly, if  $(z, w) \in A$  and  $w - z \leq 0$ , then  $\frac{w_n}{w} \geq 1$ . Therefore, for any  $(z, w) \in A$ ,  $(z_n, w_n) \not\rightarrow (0, 0)$ . Hence,  $A$  is  $f$ -invariant and points in  $A$  converge to the characteristic direction  $[1 : 1]$ , but do not converge to the origin.

#### 4. NO DOMAIN OF ATTRACTION TO THE ORIGIN IN $\mathbb{C}^2$

While some points converge to the origin under iteration, we now show that there is no (non-empty) domain of  $\mathbb{C}^2$  whose points converge to the origin. Consider any point  $(z, w) \in (\mathbb{C}^\times)^2$  that is near the origin; in particular,  $(z, w) \in B_\epsilon := \{(z, w) \in \mathbb{C}^2 \mid 0 < |z| < \epsilon, 0 < |w| < \epsilon\}$  for  $\epsilon < \frac{1}{4}$ . Then:

$$\begin{aligned} z_n = z \prod (1 - (z_j - w_j)) \rightarrow 0 &\Leftrightarrow \sum \log |1 - (z_j - w_j)| \rightarrow -\infty && \text{or } z_k - w_k = 1 \text{ some } k \\ &\Leftrightarrow \sum (-2 \operatorname{Re}(z_j - w_j) + |z_j - w_j|^2) \rightarrow -\infty && \text{or } z_k - w_k = 1 \\ (4.1) \quad &\Rightarrow \sum \operatorname{Re}(z_j - w_j) \rightarrow \infty && \text{or } z_k - w_k = 1 \\ w_n = w \prod (1 + (z_j - w_j)) \rightarrow 0 &\Leftrightarrow \sum \log |1 + (z_j - w_j)| \rightarrow -\infty && \text{or } z_l - w_l = -1 \text{ some } l \\ &\Leftrightarrow \sum (2 \operatorname{Re}(z_j - w_j) + |z_j - w_j|^2) \rightarrow -\infty && \text{or } z_l - w_l = -1 \\ (4.2) \quad &\Rightarrow \sum \operatorname{Re}(z_j - w_j) \rightarrow -\infty && \text{or } z_l - w_l = -1. \end{aligned}$$

Since  $(z, w) \in B_\epsilon$ , we know that  $z - w \neq \pm 1$ . The limits in (4.1) and (4.2) are mutually exclusive, so in order for  $(z_n, w_n) \rightarrow (0, 0)$ , we must have that  $z_j - w_j = 1$  or  $-1$  for some  $j$ .

Suppose that  $(z_n, w_n) \rightarrow (0, 0)$ . Then some iterate of  $(z, w)$  must first leave a neighborhood of the origin (to get  $|z_j - w_j| = 1$ ) and then converge along  $\mathbb{C} \times \{0\}$  or  $\{0\} \times \mathbb{C}$  since  $w_{j+1} = 0$  or  $z_{j+1} = 0$ . Consequently, the iterates of  $(z, w)$  converge along the characteristic direction  $[1 : 0]$  or  $[0 : 1]$ . However, as we already explained in §1, there is no domain of attraction along  $[1 : 0]$  or along  $[0 : 1]$ . Hence, there is no (non-empty) domain in  $\mathbb{C}^2$  whose points are attracted to the origin.

5. ADDING HIGHER DEGREE TERMS TO  $f$ 

In this section, we will see that adding higher degree terms to  $f$  can lead to the existence of a domain of attraction to the origin along its only degenerate characteristic direction  $[1 : 1]$ . For simplicity, we use the conjugation of  $f$  given in (3.2) and denoted  $\tilde{f}$ . In particular,  $\tilde{f}(x, y) = (x - y^2, y(1 - x))$  and  $[1 : 0]$  is its only degenerate characteristic direction. Let  $g$  be as in Theorem 2:

$$(5.1) \quad g(x, y) = (x - y^2 + ax^{r+1}, y(1 - x)), \quad \text{where: } \begin{cases} (1) & a \in \mathbb{C}^\times \text{ and } r \in \mathbb{N}_{\geq 3} \text{ or} \\ (2) & a \notin \mathbb{R}_{\geq 0} \text{ and } r = 2. \end{cases}$$

We will see that  $g$  and  $[1 : 0]$  satisfy the conditions in theorem [L3, Theorem A] stated below.

Before stating the theorem, we give a few relevant definitions. Near the origin, a holomorphic self-map  $f$  of  $\mathbb{C}^2$  that fixes the origin, is tangent to the identity, and  $f \neq \text{Id}$  can be written as:

$$(5.2) \quad f(z) = z + P_{k+1}(z) + P_{k+2}(z) + \cdots,$$

where  $P_j := (p_j, q_j)$  is a homogeneous polynomial of degree  $j$  and  $k+1$  is the *order* of  $f$  ( $P_{k+1} \neq 0$ ).

**Definition 4.** Let  $Q : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a homogeneous polynomial and suppose that  $Q(v) = \lambda v$  for  $v \in \mathbb{C}^n \setminus \{O\}$  and  $\lambda \in \mathbb{C}$ . The projection of  $v$  to  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$  is called a *characteristic direction*;  $[v]$  is *degenerate* if  $\lambda = 0$  and *non-degenerate* if  $\lambda \neq 0$ .

**Definition 5.** For  $f$  as in (5.2),  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$  is a *characteristic direction of degree  $s$*  if it is a characteristic direction of  $P_{k+1}, \dots, P_s$ , where  $s \geq k+1$ . In addition,  $[v]$  is *non-degenerate in degree  $r+1$*  if it is degenerate for  $P_{k+1}, \dots, P_r$  and non-degenerate for  $P_{r+1}$ , where  $r > k$ .

**Theorem A** [L3]. *Let  $g$  be a germ of a holomorphic self-map of  $\mathbb{C}^2$  that is tangent to the identity at the fixed point  $O$ , is of order  $k+1$ , and has characteristic direction  $[v] \in \mathbb{P}^1(\mathbb{C})$ . Assume  $[v]$  is:*

- (1) *a characteristic direction of degree  $s \leq \infty$ ;*
- (2) *non-degenerate of degree  $r+1$ , where  $k < r < s$ ; and*
- (3) *of order one in degree  $t+1$ , where  $k \leq t \leq r$ .*

*If  $[v]$  is transversally attracting and  $s > r + t - k$ , then there exists a domain of attraction whose points, under iteration by  $g$ , converge to  $O$  along  $[v]$ .*

In particular,  $g$  with  $[v] = [1 : 0]$  satisfies the conditions of this theorem since  $1 = k = t < r < s = \infty$  and  $[v]$  is *transversally attracting* by assumption (1) or (2) in (5.1). Hence,  $g$  has a domain of attraction to the origin along  $[1 : 0]$ . Note that  $g$  and  $[1 : 0]$  would continue to satisfy the conditions of [L3, Theorem A] if we added many different types of higher degree terms to  $g$ . In particular, we could add  $\eta$  to  $g$ , where  $\eta(x, y) = (y O((x, y)^2) + O(x^{r+2}), y O((x, y)^2) + O(x^{r+2}))$ .

Estimates for the rates of convergence of  $g^n(x, y) = (x_n, y_n)$  within the domain of attraction are given in [L3, Proposition 4.2]. In particular, for  $t = k = 1 < r$  and  $s = \infty$ :

$$x_n \lesssim n^{-\frac{1}{r}} \quad \text{and} \quad y_n \lesssim e^{-\text{Re } \beta \frac{r}{r-1} n^{\frac{r-1}{r}}},$$

where  $\beta := (-ar)^{-\frac{1}{r}}$  and the  $\frac{1}{r}$ -th root is chosen so that  $\text{Re } \beta > 0$ , which we can do when  $r > 2$  or  $r = 2$  and  $a \notin \mathbb{R}_{\geq 0}$ . If we allow  $r$  to become arbitrarily large, the growth rates approach:

$$(5.3) \quad x_n \lesssim 1 \quad \text{and} \quad y_n \lesssim e^{-n \text{Re } \beta}.$$

The domain of attraction for  $g$  given in [L3] is  $g$ -invariant for arbitrarily large  $r$ , but it is no longer invariant if  $r = \infty$ , so we cannot quite extrapolate that the growth rates in (5.3) hold when  $r = \infty$ . However, the growth rates help support the following conjecture for  $f$  since, near the origin, the behavior of  $g$  approaches the behavior of  $\tilde{f}$  as  $r$  goes to infinity.

**Conjecture 1.** *There is a non-empty,  $f$ -invariant domain in  $\mathbb{C}^2$  whose points converge to its degenerate characteristic direction  $[1 : 1]$ .*

The author can further support this conjecture from computations done using Dynamics Explorer. In particular, it appears that there is a domain with the origin in its boundary and  $\operatorname{Re} z, \operatorname{Re} w > 0$  (equivalently,  $\operatorname{Re}(x + y), \operatorname{Re}(x - y) > 0$ ) whose points converge to  $\{z = w\}$  (equivalently,  $\{y = 0\}$ ).

An interesting borderline case is when  $r = 2$  in (5.1). Then the conditions of [L3, Theorem A] are not automatically satisfied. In particular, for this case, the conditions of the theorem are not satisfied if and only if  $a \in \mathbb{R}_{\geq 0}$ . We use  $h$  to represent this special case (assume  $a \neq 0$  so  $h \neq \tilde{f}$ ):

$$(5.4) \quad h(x, y) = (x - y^2 + ax^3, y(1 - x)), \quad \text{for any } a \in \mathbb{R}_{>0}.$$

Theorem 3 is about the dynamics of  $h$  and it is proven later on in this section.

Before proving Theorem 3, we compare  $g, h$  and  $f$  graphically. Converting  $g$  and  $h$  to  $(z, w)$ -coordinates we get:

$$\tilde{g}(z, w) = \left( z(1 - (z - w)) + \frac{a}{2}(z + w)^{r+1}, w(1 + (z - w)) + \frac{a}{2}(z + w)^{r+1} \right).$$

Notice that:  $\tilde{g} = f$  when  $a = 0$ ,  $\tilde{g}$  corresponds to  $h$  when  $r = 2$  and  $a \in \mathbb{R}_{>0}$ , and  $\tilde{g}$  corresponds to  $g$  otherwise. Since  $\tilde{g}$  arises from  $f$  by adding higher degree terms only, the types of characteristic directions for both maps are the same. In particular,  $f$  and  $\tilde{g}$  have  $[1 : 0], [0 : 1]$  as non-attracting, non-degenerate characteristic directions and  $[1 : 1]$  as a degenerate characteristic direction. Consequently, the only hope for  $\tilde{g}$  to have a domain of attraction to the origin along a direction is to have it along  $[1 : 1]$ . Below are dynamical pictures in  $\mathbb{R}^2$  of  $\tilde{g}$  with  $r = 2$  and  $a = \pm 0.1, 0$ .

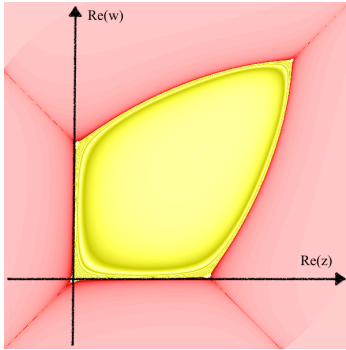


FIGURE 3.  
 $a = -0.1 \Rightarrow g$

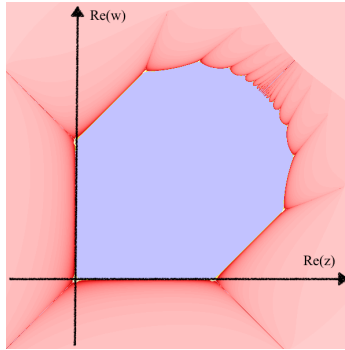


FIGURE 4.  
 $a = 0 \Rightarrow f$

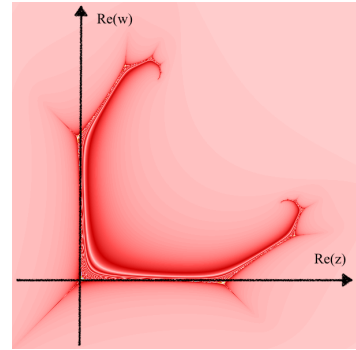


FIGURE 5.  
 $a = 0.1 \Rightarrow h$

These three figures show the dynamics of  $\tilde{g}$  in the window  $[-0.5, 2]^2 \subset \mathbb{R}^2$  with  $r = 2$ . Points are:

- yellow if they converge to  $O$ ,
- blue if they converge to  $[1 : 1]$ , but not to  $O$ , and
- red if they leave a neighborhood of  $O$  (in particular,  $|(z_j, w_j)| > 5$  for some  $j$ ).

The different shades of a given color represent how quickly a point converges or diverges. Note that some curves that converge to  $[1 : 1]$  appear red because nearby points that do not lie on those curves diverge from  $[1 : 1]$  and  $O$ ; a single curve is too thin to appear blue on its own.

Notice that  $\tilde{g}$  sends  $\{z = w\}$  to  $\{z_1 = w_1 = z(1 + a(2z)^r)\}$ , so  $\{z = w\}$  is  $\tilde{g}$ -invariant, but not pointwise fixed (unless  $a = 0$  and so  $f = \tilde{g}$ ). Restricting to  $\mathbb{R}^2$  and  $r = 2$ , it is clear that points

on  $\{z = w\}$  converge to the origin for sufficiently small  $|z|$  exactly when  $a < 0$ . Expanding to  $\{z = w\} \subset \mathbb{C}^2$  and  $r \geq 2$ , there is an open subset of  $\{z = w\}$  that converges to the origin as long as  $a \neq 0$  by the Leau-Fatou Flower Theorem [A1].

Now we focus on the case when  $r = 2$  and  $a \in \mathbb{R}_{>0}$ , as in (5.4) and Figure 5.

**Theorem 3.** *Let  $h(x, y) = (x - y^2 + ax^3, y - xy)$ . If  $a \in \mathbb{R}_{>0}$ , then  $h$  has no domain of attraction to the origin along any direction.*

**Proof.** First of all, by [H1, Proposition 1.3], if an orbit of a point converges to the origin along a direction, that direction must be a characteristic direction. Since  $h$ , up through degree 2, is conjugate to  $f$  from Theorem 1 it has the same types of characteristic directions as  $f$ . In particular,  $[1 : 1]$  and  $[1 : -1]$  are both non-degenerate and non-attracting characteristic directions (for each, the real part of its director is negative). Hence,  $h$  has no domain of attraction along either of those directions. The remaining characteristic direction,  $[1 : 0]$ , is degenerate.

Suppose, for contradiction, that there exists an  $\Omega \subset \mathbb{C}^2$  that is a domain of attraction to the origin along  $[1 : 0]$ . Then  $\Omega$  is an open set with  $O \in \partial\Omega$  and for any  $(x, y) \in \Omega$ , the iterates  $x_n, y_n$ , and  $\frac{y_n}{x_n} \rightarrow 0$ . If  $y = 0$ , then  $h(x, y) = (x(1 + ax^2), 0)$  and the Leau-Fatou Flower Theorem tells us that there are domains in  $\mathbb{C} \times \{0\}$  whose points are attracted to the origin. That gives us open subsets of  $\mathbb{C} \times \{0\}$  that converge to the origin (trivially) along  $[1 : 0]$ . Let  $u = \frac{y}{x}$  and  $v = \frac{y}{x^2} = \frac{u}{x}$ . Then:

$$\begin{aligned}
 (5.5) \quad x_1 &= x(1 + x[ax - u^2]) &= x(1 + x^2[a - xv^2]) \\
 u_1 &= u \frac{1 - x}{1 + x[ax - u^2]} &= u(1 - x[1 + (ax - u^2)] + x^2 O(x, u^2)) \\
 & &= u(1 - x + O(x^2, xu^2)) \\
 v_1 &= v \frac{1 - x}{1 + x[ax - u^2][2 + x(ax - u^2)]} &= v(1 - x[1 + 2(ax - u^2)] + x^2 O(x, u^2)) \\
 & &= v(1 - x + O(x^2, xu^2))
 \end{aligned}$$

where  $(x, y) \in \Omega$  and we assume that  $|x|, |u| \ll 1$  so that we can use the geometric series to get the equalities on the right. We can use that  $|x|$  and  $|u|$  are small since we are working near the origin and  $(x, y) \in \Omega$  so  $x_n \rightarrow 0$  and  $u_n \rightarrow 0$ . The  $n$ th-iterates of  $x, u$ , and  $v$  are:

$$\begin{aligned}
 (5.6) \quad x_n &= x \prod_{j=0}^{n-1} (1 + x_j(ax_j - u_j^2)) = x \prod_{j=0}^{n-1} (1 + x_j^2(a - x_j v_j^2)) \\
 u_n &= u \prod_{j=0}^{n-1} (1 - x_j [1 + (ax_j - u_j^2) + x_j O(x_j, u_j^2)]) = u \prod_{j=0}^{n-1} (1 - x_j + O(x_j^2, x_j u_j^2)) \\
 v_n &= v \prod_{j=0}^{n-1} (1 - x_j [1 + 2(ax_j - u_j^2) + x_j O(x_j, u_j^2)]) = v \prod_{j=0}^{n-1} (1 - x_j + O(x_j^2, x_j u_j^2)),
 \end{aligned}$$

where  $(x, y) \in \Omega$  and we use  $|x_j|, |u_j| \ll 1$  as we did, with  $j = 0$ , to get (5.5). If  $y \neq 0$  and  $y_n \rightarrow 0$ , then  $\log |y_n| \rightarrow -\infty$  implies:

$$(5.7) \quad \sum \log |1 - x_j| \rightarrow -\infty \Rightarrow \sum \log(1 - 2 \operatorname{Re} x_j + |x_j|^2) \rightarrow -\infty \Rightarrow \sum \operatorname{Re} x_j \rightarrow \infty.$$

Notice that the final expressions for  $u_n$  and  $v_n$  in (5.6) appear the same. For  $|x|$  and  $|u|$  sufficiently small, the big-O in  $u_n$  and  $v_n$  effectively act the same and so  $u_n \rightarrow 0$  implies  $v_n \rightarrow 0$ .



Now we show  $\Omega$  cannot exist when  $a > 0$ . Let  $t = \frac{1}{x}$ . Using the expression for  $x_1$  in (5.5), we get:

$$t_1 = \frac{t}{1 + \frac{1}{t^2}(a - \frac{v^2}{t})} = t - \frac{a}{t} + O\left(\frac{v^2}{t^2}, \frac{1}{t^3}\right)$$

$$t_n = t - a \sum_j \frac{1}{t_j} \left(1 + O\left(\frac{v_j^2}{t_j}, \frac{1}{t_j^2}\right)\right)$$

Since  $|v_j|$  and  $|t_j|^{-1} = |x_j|$  are small and go to 0 as  $j \rightarrow \infty$ , the big-O terms in  $t_n$  are much smaller than 1. Hence,

$$\operatorname{Re} t_n \approx \operatorname{Re} t - a \sum_j \operatorname{Re} \frac{1}{t_j} = \operatorname{Re} t - a \sum_j \operatorname{Re} x_j \rightarrow -\infty,$$

where we get the limit on the right from (5.7) and our assumption that  $a > 0$ . This implies that for some  $N > 1$ , which can depend on  $(x, y)$ ,  $\operatorname{Re} t_n < 0$  for all  $n > N$ . Note that:

$$\operatorname{Re} x_j = \operatorname{Re} \frac{1}{t_j} = \frac{1}{|t_j|^2} \operatorname{Re} t_j = |x_j|^2 \operatorname{Re} t_j.$$

Combining this and that  $y_n \rightarrow 0$  implies  $\sum \operatorname{Re} x_j \rightarrow \infty$  from (5.7):

$$\sum_{j=1}^{\infty} \operatorname{Re} x_j = \sum_{j=1}^N \operatorname{Re} x_j + \sum_{j=N+1}^{\infty} |x_j|^2 \operatorname{Re} t_j = (\text{finite number}) + (\text{sum of negative numbers}) \not\rightarrow +\infty.$$

This is a contradiction. Hence, there is no domain of attraction to the origin along  $[1 : 0]$ .  $\square$

**Conjecture 2.** *For  $h$  is as in Theorem 3, there is no domain (in  $\mathbb{C}^2$ ) whose points converge to  $O$ .*

We showed in Theorem 3 that if such a domain exists, none of its points can converge along a direction. For the example in Figure 5, we see that if such a domain exists, it should not intersect  $\mathbb{R}^2 \setminus \{O\}$ . The author can further support this conjecture from numerous computations done using Dynamics Explorer, where  $a > 0$  was varied and  $z, w$  were also considered in  $\mathbb{C}$  (not just  $\mathbb{R}$ ).

## 6. SUMMARY

The main example in this paper,

$$f(z, w) = (z[1 - (z - w)], w[1 + (z - w)]),$$

has no domain in  $\mathbb{C}^2$  whose points are attracted to the origin. The characteristic direction  $[1 : 1]$ , which corresponds to  $\{z = w\}$ , attracts points in an open set  $A \subset \mathbb{R}^2$  to itself, but not to the origin. The author, in Conjecture 1, hypothesizes that there exists a domain in  $\mathbb{C}^2$  that is attracted to  $\{z = w\}$ . This analysis of  $f$ , which is summarized in Theorem 1, provides a fairly detailed description of the dynamics of  $f$  in a full neighborhood of the origin. By adding higher degree terms to  $f$ , as we did in §5, we can significantly alter the behavior of points along the degenerate characteristic direction. Sometimes, but not always, the addition of higher degree terms can lead to stronger convergence than in  $f$ ; in particular, the existence of a domain of attraction.

The dynamical behavior of the main example in  $\mathbb{R}^2$  is summarized in the following Figure 6. The region  $A$  is the triangle bounded by the axes and the purple line. The dynamics in  $A$  is of particular interest as it is the first example, as far as the author knows, of a set whose points: (1) can remain arbitrarily close to the origin under iteration, (2) are not all fixed, (3) converge to a characteristic direction, and (4) do not converge to the origin.

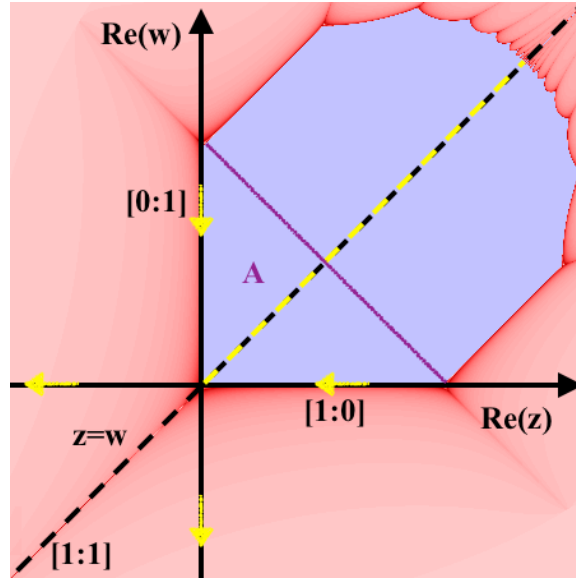


FIGURE 6.

The dynamics of  $f$  in  $\mathbb{R}^2$ . Yellow indicates direction of movement. For  $(z, w) \in \mathbb{R}^2$ :

- if  $z = 0$  and  $0 < w < 1$ , then  $(z_n, w_n) \rightarrow (0, 0)$  along  $[0 : 1]$ ;
- if  $w = 0$  and  $0 < z < 1$ , then  $(z_n, w_n) \rightarrow (0, 0)$  along  $[1 : 0]$ ;  $f$
- if  $(z, w) \in A$  (or in blue region), then  $(z_n, w_n) \not\rightarrow (0, 0)$  and  $[z_n : w_n] \rightarrow [1 : 1]$ ; and
- if  $(z, w)$  in red region, then  $(z_n, w_n) \not\rightarrow (0, 0)$  and  $[z_n : w_n] \not\rightarrow [1 : 1]$ .

*This picture was drawn using a profile written for the program Dynamics Explorer.*

The domain pictured is:  $[-0.8, 1.6] \times [-0.8, 1.6]$ .

## REFERENCES

- [A1] M. Abate. *Holomorphic classification of 2-dimensional quadratic maps tangent to the identity*. Surikaiseikikenkyusho Kokyuroku, **1447** (2005), 1-14.
- [A2] M. Abate. *Discrete holomorphic local dynamical systems*. “Holomorphic dynamics,” Lect. Notes in Math., Springer (2009).
- [AR] M. Arizzi and J. Raissy. *On Écalle-Hakim’s theorems in holomorphic dynamics*. Proceedings of Frontiers in Complex Dynamics (2011), [arXiv:1106.1710v2](#).
- [H1] M. Hakim. *Transformations tangent to the identity. Stable pieces of manifolds*. Preprint (1997).
- [H2] M. Hakim. *Analytic transformations of  $(\mathbb{C}^p, 0)$  tangent to the identity*. Duke Mathematical Journal, **92** (1998), 403-428.
- [L1] S. Lapan. *On the existence of attracting domains for maps tangent to the identity*. Ph.D. Thesis, U. of M. (2013).
- [L2] S. Lapan. *Attracting domains of maps tangent to the identity whose only characteristic direction is non-degenerate*. Int. J. Math., **24** (2013).
- [L3] S. Lapan. *Domain of attraction for maps tangent to the identity in  $\mathbb{C}^2$  with characteristic direction of higher degree*. Preprint (2015). [arXiv:1501.00244v2](#).
- [Ro1] F. Rong. *Absolutely isolated singularities of holomorphic maps of  $\mathbb{C}^n$  tangent to the identity*. Pacific J. Math. **246** (2010), 421-433.
- [Ro2] F. Rong. *Quasi-parabolic analytic transformations of  $\mathbb{C}^n$ . Parabolic manifolds*. Ark. Mat., **48** (2010), 361-370.
- [V] L. Vivas. *Degenerate characteristic directions for maps tangent to the identity*. [arXiv:1106.1471](#) (December 9, 2011). To appear in Indiana Journal of Mathematics.
- [W1] B. Weickert. *Automorphisms of  $\mathbb{C}^2$  tangent to the identity*. Ph.D. Thesis, University of Michigan (1997).
- [W2] B. Weickert. *Attracting basins for automorphisms of  $\mathbb{C}^2$* . Invent. Math. **132** (1998), 581-605.